Research Summary: Phase Retrieval

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Phase Retrieval Problem – Background

The phase retrieval problem aims to reconstruct the phase of a complex signal based on its modulus measurement.
When reconstructing the signal, additional prior information is normally used.

- Fienup's Hybrid Input Output(HIO) algorithm performs extremely robust and is a de-facto choice in practice.
- Generalized Expectation Consistent(GEC) algorithm is widely believed in the statistical physics community to be an "optimal" polynomial time algorithm.

Since these algorithms are proposed in different communities, no direct comparison between them had been conducted before.

Phase Retrieval Problem – System Model

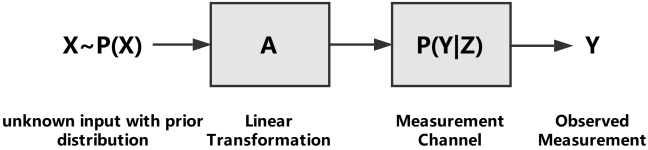

Figure: The generalized linear model

Phase Retrieval Problem – System Model

Unknown input signal $\boldsymbol{x}$ is a i.i.d complex gaussian signal with prior distribution:

$$\boldsymbol{x} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{I}) \tag{1}$$

$\boldsymbol{A} \in \mathbb{R}^{M \times N}$ is a normalized column orthogonal sensing matrix. (measurement ratio $\delta = M/N$) The output of the sensing matrix $\boldsymbol{z}$ follows:

$$\boldsymbol{z} = \boldsymbol{A}\boldsymbol{x} \tag{2}$$

$\boldsymbol{z}$ is measured through a noiseless channel and the output $\boldsymbol{y}$ is the modulus of $\boldsymbol{z}$:

$$\boldsymbol{y} = |\boldsymbol{z}| \tag{3}$$

We aim to reconstruct the signal $\boldsymbol{x}$ using the measurement $\boldsymbol{y}$.

Phase Retrieval Problem – HIO algorithm

Fienup introduced the HIO algorithm in 1982: This method considered two different kinds of constraints:

- The support constraint:($\boldsymbol{z}^{\boldsymbol{t}}$ is the reconstructed $\boldsymbol{z}$ in the $\boldsymbol{t}$ iteration)

$$\boldsymbol{z}^{\boldsymbol{t}} \in \boldsymbol{range}(\boldsymbol{A}) \tag{4}$$

- The modulus constraint:

$$|\boldsymbol{z}^{\boldsymbol{t}}| = \boldsymbol{y} \tag{5}$$

Using an iterative method to find the signal $\boldsymbol{x}$ which satisfies:

$$\boldsymbol{x} = \boldsymbol{Sup} \cap \boldsymbol{Mod} \tag{6}$$

Phase Retrieval Problem – HIO algorithm

We can derive two projection onto ***Sup*** and ***Mod***:

$$P_S = A(A^H A)^{-1} A^H z \quad (7)$$

$$P_M = y. * arg(z) \quad (8)$$

$$arg(z) = \frac{z}{|z|} \quad (9)$$

The iteration algorithm can be expressed as:

$$z^{t+1} = z^t + \beta P_M(2P_S(z^t) - z^t) - P_S(z^t) \quad (10)$$

Where β is a damping parameter and it usually equals to 0.5.

Phase Retrieval Problem – GEC algorithm

The GEC algorithm include two blocks and transfer extrinsic information between two blocks using turbo method.

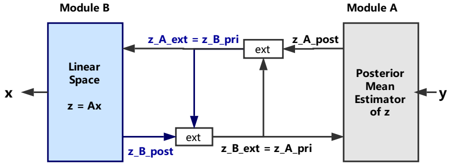

Figure: The block diagram of GEC algorithm

- Block A: calculate the posteriori mean and variance of the signal using the prior distribution and measurement(MMSE Estimation)
- Block B: constrain the signal into the linear space.

Performance of Algorithm with the Change of Initialization

The performance of the algorithm is measured using the minimal δ that makes the algorithm converge. We build the initialized signal

$$\hat{\boldsymbol{z}} = \frac{\frac{1}{\delta}}{\frac{1}{\delta} + v_0} * (\boldsymbol{z} + \sqrt{v_0}\mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})) \qquad (11)$$

The change of v_0 denotes the "distance" between the initialization and original signal.

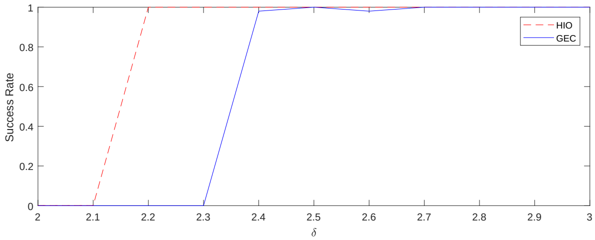

Figure: The comparison of performance between two algorithms under $v_0 = 1$

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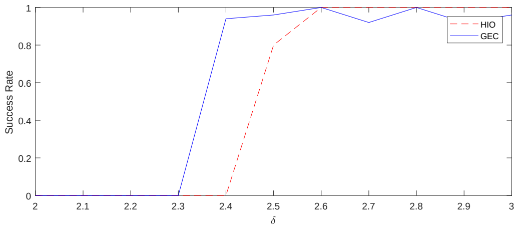

Figure: The comparison of performance between two algorithms under $v_0 = 100$

Performance of Algorithm with the Change of Initialization
(i) HIO algorithm is sensitive to the initialization, while for GEC algorithm the successfully reconstructed δ doesn't vary too much if the initialization is **postively correlated** with original signal.
(ii) If the initialization is good enough, then HIO is better than GEC; and if the initialization is not so good, then GEC is better.
(iii) The **spectral initialization** is usually used in practice. The further result shows that the GEC algorithm perform better than HIO algorithm using spectral initialization, which shows that spectral initialization may not be the optimal initialization in the Gaussian circumstance.

For GEC algorithm, there exist a mathematical approach to predict the performance of the algorithm under different circumstances. The two blocks are treated separately. The input and output variances of each block are tested.

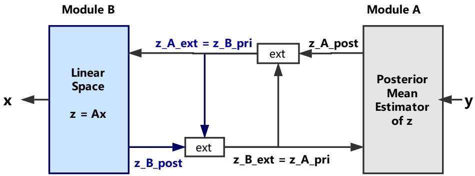

Figure: The block diagram of GEC algorithm

Phase Retrieval Problem – State Evolution

The normalized input & output curves of two blocks are plotted as below. The convergence of the algorithm and estimated MSE can be predicted from the curve.

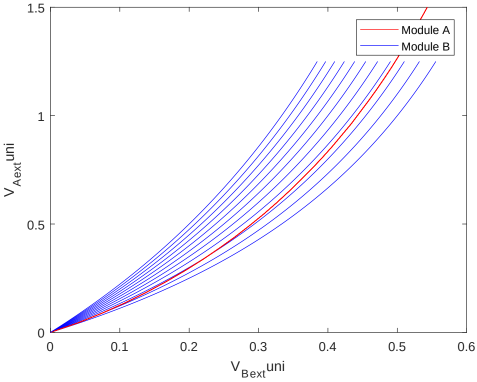

Figure: The state evolution curves

Phase Retrieval Problem – State Evolution

A theorem is proposed and proved regarding the successfully reconstruction boundary

Theorem

The estimate generated by GEC algorithm is asymptotically orthogonal to the true signal if the measure ratio is smaller than two.

$$\lim_{t \to +\infty} \mathbb{E}\left[\frac{\langle \hat{\boldsymbol{x}}^t, \boldsymbol{x} \rangle}{\|\hat{\boldsymbol{x}}^t\|_2 \|\boldsymbol{x}\|_2}\right] = 0 \tag{13}$$

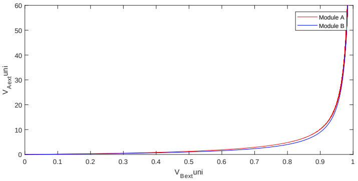

Figure: The relationship between two models when $\delta = 2$

The analytical theorem correspond to the numerical result:
When $\delta < 2$:

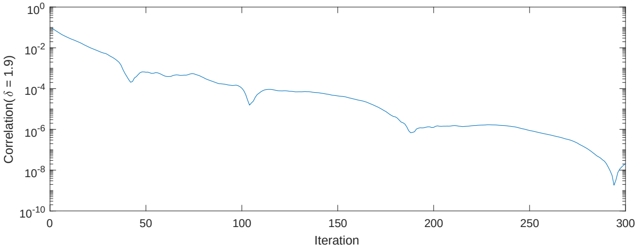

Figure: The correlation between the reconstructed signal and the original signal when $\delta = 1.9$

Phase Retrieval Problem – State Evolution

When $\delta > 2$:

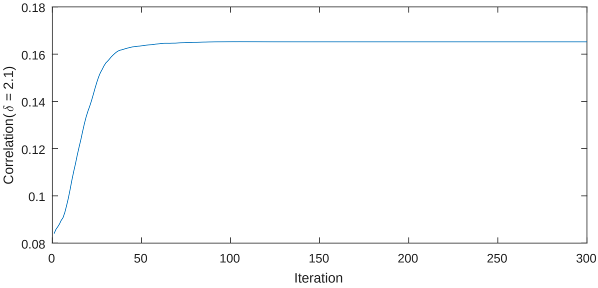

Figure: The correlation between the reconstructed signal and the original signal when $\delta = 2.1$